Problem 1.59

Check the divergence theorem for the function

$$\mathbf{v} = r^2 \sin\theta \,\hat{\mathbf{r}} + 4r^2 \cos\theta \,\hat{\boldsymbol{\theta}} + r^2 \tan\theta \,\hat{\boldsymbol{\phi}},$$

using the volume of the "ice-cream cone" shown in Fig. 1.52 (the top surface is spherical, with radius R and centered at the origin). [*Answer:* $(\pi R^4/12)(2\pi + 3\sqrt{3})$.]

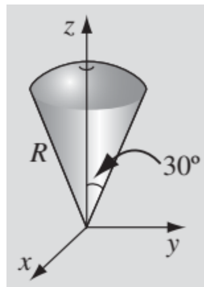

Fig. 1.52

Solution

In spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis, the divergence of a vector function is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

For the given function, it evaluates to

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(r^2\sin\theta)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[(4r^2\cos\theta)\sin\theta] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta) \\
&= \frac{1}{r^2}(4r^3\sin\theta) + \frac{1}{r\sin\theta}[(-4r^2\sin\theta)\sin\theta + (4r^2\cos\theta)\cos\theta] + \frac{1}{r\sin\theta}(0) \\
&= \cancel{4r\sin\theta} + 4r\left(\cancel{-\sin\theta} + \frac{\cos^2\theta}{\sin\theta}\right) \\
&= 4r\frac{\cos^2\theta}{\sin\theta}.
\end{aligned}$$

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla \cdot \mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

Here D is the 30° ice-cream cone shown in Fig. 1.52.

Calculate the left side.

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{v}\,dV &= \int_0^{\pi/6}\int_0^{2\pi}\int_0^R \left(4r\frac{\cos^2\theta}{\sin\theta}\right)(r^2\sin\theta\,dr\,d\phi\,d\theta) \\
&= \int_0^{\pi/6}\int_0^{2\pi}\int_0^R 4r^3\cos^2\theta\,dr\,d\phi\,d\theta \\
&= 4\left(\int_0^{\pi/6}\cos^2\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R r^3\,dr\right) \\
&= 4\left[\int_0^{\pi/6}\frac{1}{2}(1+\cos 2\theta)\,d\theta\right]\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R r^3\,dr\right) \\
&= 4\left[\frac{1}{24}(2\pi+3\sqrt{3})\right](2\pi)\left(\frac{R^4}{4}\right) \\
&= \frac{\pi R^4}{12}(2\pi+3\sqrt{3})
\end{aligned}$$

Let the conical surface be S_1, and let the spherical surface be S_2. Noting that the direction of each area element is the outward unit vector to the surface, calculate the right side.

$$\begin{aligned}
\oiint_{\text{bdy } D}\mathbf{v}\cdot d\mathbf{S} &= \iint_{S_1}\mathbf{v}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{v}\cdot d\mathbf{S} \\
&= \int_0^{2\pi}\int_0^R (r^2\sin\theta\,\hat{\mathbf{r}} + 4r^2\cos\theta\,\hat{\boldsymbol{\theta}} + r^2\tan\theta\,\hat{\boldsymbol{\phi}})\Big|_{\theta=\pi/6}\cdot\left(\hat{\boldsymbol{\theta}}r\sin\frac{\pi}{6}\,dr\,d\phi\right) \\
&\quad + \int_0^{\pi/6}\int_0^{2\pi}(r^2\sin\theta\,\hat{\mathbf{r}} + 4r^2\cos\theta\,\hat{\boldsymbol{\theta}} + r^2\tan\theta\,\hat{\boldsymbol{\phi}})\Big|_{r=R}\cdot(\hat{\mathbf{r}}R^2\sin\theta\,d\phi\,d\theta) \\
&= \int_0^{2\pi}\int_0^R 4r^3\sin\frac{\pi}{6}\cos\frac{\pi}{6}\,dr\,d\phi + \int_0^{\pi/6}\int_0^{2\pi}R^4\sin^2\theta\,d\phi\,d\theta \\
&= \left(\sin\frac{\pi}{6}\cos\frac{\pi}{6}\right)\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R 4r^3\,dr\right) + R^4\left(\int_0^{\pi/6}\sin^2\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right) \\
&= \left(\sin\frac{\pi}{6}\cos\frac{\pi}{6}\right)\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R 4r^3\,dr\right) + R^4\left[\int_0^{\pi/6}\frac{1}{2}(1-\cos 2\theta)\,d\theta\right]\left(\int_0^{2\pi}d\phi\right) \\
&= \left(\frac{\sqrt{3}}{4}\right)(2\pi)(R^4) + R^4\left[\frac{1}{24}(2\pi-3\sqrt{3})\right](2\pi) \\
&= \frac{\pi R^4}{12}(2\pi+3\sqrt{3})
\end{aligned}$$